A note on the eigenvalues for equivariant maps of the SU(2) sigma-model

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Abstract

We numerically calculate the first few eigenvalues of the perturbations of self-similar solutions of the spherically symmetric co-rotational SU(2) sigma-model on Minkowski space.

1   Introduction

In the recent past considerable interest focused on the study of nonlinear evolution equations that show "blow-up" for solutions in a finite time. Prominent candidates are self-similar solutions which were found to play an essential role in critical collapse phenomena. Critical solutions have the property that they lie at the boundary of two possible end states of dynamical evolutions: Typically, sufficiently large initial data will lead to the formation of some kind of singularity, while small data disperse. By fine-tuning the initial data to this threshold, the dynamics shows universal behaviour which is governed by a "critical solution". These solutions are characterized by the existence of a single unstable mode in linear perturbation analysis. While it is relatively easy to obtain the eigenvalue of the unstable mode by numerical analysis, to calculate the stable ones may be more subtle. However, the knowledge of the stable modes is important because they are responsible for the process of how fine-tuned data are attracted toward the critical solution.

In this note we study equivariant maps of the SU(2) sigma-model (wave maps) from 3+1 dimensional Minkowski spacetime into the three-sphere. This system admits a discrete sequence of self-similar solutions and has been studied as a toy model for critical collapse behaviour [2]. Linear perturbation analysis shows that the "ground state" is stable, while the higher excitations are unstable. The first excitation acts as the critical solution between collapse to the ground state and dispersion. Bizoń [4] has given a beautiful algorithm how to obtain analytically all eigenvalues for the linear perturbations of the ground state. However, for the higher excitations, because they are known only numerically, one has to rely on numerical tools.

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We apply two different numerical methods to calculate the first few eigenvalues of the ground state and the first excitation i.e. the critical solution. One method is by shooting and matching an the other by time evolution. Our analysis shows that the "shooting" method, frequently applied for finding the largest eigenvalues may fail for the smaller values.

For the ground state our results confirm the analytical results. This is important because, as pointed out by Bizoń, the eigenvalue problem is not standard. Our results show that the analyticity condition for the singular boundary value problem chosen by Bizoń are in accordance with the time evolution of the system. In addition we find for the critical solution, besides of the unstable and the gauge mode, the first two stable eigenvalues.

2 The SU(2) σ-model

2.1 Definition of the model

A smooth mapping $U : (M, g) \rightarrow (S^3, G)$ from a spacetime M with metric g into the three-sphere is called SU(2) σ-model if it is a critical point of the action

$$\int_M g^{ab} \partial_a U^A \partial_b U^B G_{AB}. \tag{1}$$

Let M be Minkowski spacetime. We use polar coordinates (t, r, θ, φ) on M and standard coordinates (u, Θ, Φ) on S^3. In these coordinates the metrics are given by

$$g = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)$$

$$G = du^2 + \sin^2(u)(d\Theta^2 + \sin^2(\Theta)d\Phi^2) \tag{2}$$

We restrict ourselves to maps of the form

$$U(t, r, \theta, \varphi) = (u(t, r), \theta, \varphi) \tag{3}$$

which are called spherically symmetric and co-rotational. Under these assumptions the Euler-Lagrange equations associated with the action (1) reduce to the single semilinear wave equation

$$u_{tt} - u_{rr} - \frac{2}{r}u_r + \frac{\sin(2u)}{r^2} = 0. \tag{4}$$

2.2 Self-similar solutions

A self-similar solution of eq. (4) is a solution of the form

$$u(t, r) = f\left(\frac{r}{T - t}\right) \tag{5}$$

for some constant $T > 0$. We substitute the ansatz (5) into eq. (4) and obtain

$$f'' + \frac{2}{\rho}f' - \frac{\sin(2f)}{\rho^2(1 - \rho^2)} = 0 \tag{6}$$

where $\rho := \frac{r}{T-t}$ and $' := \frac{d}{d\rho}$. Note that the domain $\rho \in [0, 1]$ is the backward lightcone of the point $(T, 0)$ which is called the *singularity* because $u_r(t, 0)$ grows unbounded for $t \rightarrow T-$. Therefore T is called the *blow-up*

time. Eq. (6) has two singular points ($\rho = 0, 1$) and therefore we have to require the boundary conditions

$$f(0) = 0 \text{ and } f(1) = \frac{\pi}{2} \tag{7}$$

to ensure smoothness of solutions. Bizoń [1] has shown existence of a countable family $\{f_0, f_1, \ldots\}$ of smooth solutions of eq. (6) which satisfy the boundary conditions (7). The so-called *ground state* f_0 is known in closed form and given by

$$f_0(\rho) = 2 \arctan \rho. \tag{8}$$

The *higher excitations* $f_1, f_2, \ldots$ can be constructed numerically using a standard shooting technique (cf. [2]).

2.3 Perturbations

We are interested in linear perturbations around the solutions f_0 and f_1 which play a crucial role in dynamical evolution. We introduce so-called adapted coordinates (τ, ρ) defined by

$$\tau = -\log(T - t) \text{ and } \rho = \frac{r}{T - t}. \tag{9}$$

These coordinates cover the domain $t < T$ (the singularity is shifted to $\tau \to \infty$). In the new coordinates eq. (4) becomes

$$u_{\tau\tau} + 2\rho u_{\tau\rho} - (1 - \rho^2)u_{\rho\rho} + u_\tau - 2\frac{1 - \rho^2}{\rho}u_\rho + \frac{\sin(2u)}{\rho^2} = 0. \tag{10}$$

It is important to note that the class of self-similar solutions which blow up as $t \to T-$ are static (i.e. independent of τ) in these coordinates. Inserting

$$u(\tau, \rho) = f_n(\rho) + w(\tau, \rho) \tag{11}$$

into eq. (10) and neglecting terms of order w^2 we obtain a time evolution equation for linear perturbations around the n-th self-similar solution f_n.

$$w_{\tau\tau} + 2\rho w_{\tau\rho} - (1 - \rho^2)w_{\rho\rho} + w_\tau - 2\frac{1 - \rho^2}{\rho}w_\rho + \frac{2\cos(2f_n)}{\rho^2}w = 0 \tag{12}$$

To ensure regularity we have to require the boundary conditions

$$w(\tau, 0) \equiv 0 \text{ and } w_{\rho\rho}(\tau, 0) \equiv 0. \tag{13}$$

We are looking for perturbations of the form

$$w(\tau, \rho) = e^{\lambda\tau}v_\lambda(\rho) \tag{14}$$

where λ may be complex in general. λ is said to be an *eigenvalue* and v_λ an *eigenmode* or *eigenfunction*. Like in the finite-dimensional case one expects that these eigenmodes determine the dynamics in the neighbourhood of the solution f_n in phase space.

To obtain an ordinary differential equation for the eigenmodes v_λ we insert ansatz (14) into eq. (12).

$$v_\lambda''(\rho) + 2\frac{(\lambda + 1)\rho^2 - 1}{\rho(\rho + 1)(\rho - 1)}v_\lambda'(\rho) + \frac{\lambda(\lambda + 1)\rho^2 + 2\cos(2f_n(\rho))}{\rho^2(\rho + 1)(\rho - 1)}v_\lambda(\rho) = 0. \tag{15}$$

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This equation has again two singular points. The regularity conditions are given by

$$v_\lambda(0) = 0, \ v_\lambda'(0) = a$$

$$v_\lambda(1) = 1, \ v_\lambda'(1) = -\frac{\lambda^2 + \lambda - 2}{2\lambda} \tag{16}$$

where $a \in \mathbb{R}$ is a free parameter and we have normalized the eigenfunctions to have $v_\lambda(1) = 1$.

3 Numerical calculation of eigenvalues

3.1 Calculating eigenvalues by time evolution

We consider the Cauchy problem eq. (12) together with initial data $w(0, \rho) = f(\rho)$ and $w_\tau(0, \rho) = g(\rho)$ inside the backward lightcone of the singularity, that is for $\rho \in [0, 1]$. By setting $u_1 = w$, $u_2 = w_\tau$, $u_3 = w_\rho$ and $u = (u_1, u_2, u_3)$ this problem can be written as a first order system

$$\frac{d}{d\tau}u = Lu \tag{17}$$

with initial conditions $u_1(0, \rho) = f(\rho)$, $u_2(0, \rho) = g(\rho)$ and $u_3(0, \rho) = f'(\rho)$. L is the linear differential operator following from eq. (12). If the Cauchy problem is well-posed one can introduce a one-parameter family of bounded linear operators $S(\tau)$ for $\tau \geq 0$ such that the solution to initial data $u(0, \cdot)$ is given by

$$u(\tau, \cdot) = S(\tau)u(0, \cdot). \tag{18}$$

We are looking for special solutions defined by eq. (14). Inserting this ansatz into eq. (18) one obtains

$$S(\tau)(v_\lambda, \lambda v_\lambda, v_\lambda') = e^{\lambda\tau}(v_\lambda, \lambda v_\lambda, v_\lambda') \tag{19}$$

for an eigenmode v_λ with eigenvalue λ. For fixed τ we define an inner product by

$$\langle u(\tau, \cdot)|v(\tau, \cdot)\rangle := \int_0^1 \sum_{j=1}^3 u_j(\tau, \rho)\overline{v_j(\tau, \rho)}d\rho \tag{20}$$

and a norm by

$$\|u(\tau, \cdot)\| := \sqrt{\langle u(\tau, \cdot)|u(\tau, \cdot)\rangle}. \tag{21}$$

If v_λ is an eigenmode with eigenvalue λ then we have

$$\|S(\tau)(v_\lambda, \lambda v_\lambda, v_\lambda')\| = |e^{\lambda\tau}|\|(v_\lambda, \lambda v_\lambda, v_\lambda')\| = Ce^{\mathrm{Re}\lambda\tau} \tag{22}$$

where C is constant. Now we make the following assumptions:

- The eigenvalues $\{\lambda_0, \lambda_1, \ldots\}$ form a discrete set

- Arbitrary initial data can be decomposed into a sum of eigenmodes

Then we can write (real-valued) initial data $u(0, \cdot)$ as

$$u(0, \rho) = \sum_{k=0}^\infty (c_k\varphi_k(\rho) + \overline{c_k\varphi_k(\rho)}) = 2\sum_{k=0}^\infty \mathrm{Re}(c_k\varphi_k(\rho)) \tag{23}$$

where $\varphi_k := (v_{\lambda_k}, \lambda_k v_{\lambda_k}, v_{\lambda_k}')$ and v_{λ_k} is an eigenmode with eigenvalue λ_k.

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For simplicity we write $\mu_k := \mathrm{Re}\lambda_k$, $\omega_k := \mathrm{Im}\lambda_k$ and without loss of generality we assume that $\mu_0 \geq \mu_1 \geq \ldots$. Applying the time evolution operator $S(\tau)$ to initial data $u(0, \cdot)$ yields

$$S(\tau)u(0, \cdot) = 2\sum_{k=0}^{\infty} \mathrm{Re}(c_k e^{\lambda_k \tau} \varphi_k) = 2\sum_{k=0}^{\infty} e^{\mu_k \tau} \mathrm{Re}(c_k e^{i\omega_k \tau} \varphi_k). \qquad (24)$$

So for large τ and $\mu_0 > \mu_1$ we have

$$\|S(\tau)u(0, \cdot)\| \sim e^{\mu_0 \tau} g(\tau) \qquad (25)$$

where $g(\tau) := \|\mathrm{Re}(c_0 e^{i\omega_0 \tau} \varphi_0)\|$. If there are more eigenvalues with the same real part μ_0 then the function g in (25) is more complicated but the result is essentially the same.

For computing the real part μ_0 of the dominant eigenvalue one simply evolves arbitrary initial data and calculates the norm (21). As long as the expansion coefficient c_0 in (23) does not vanish, the logarithm of the norm will behave as $\mu_0 \tau + \log(g(\tau))$ for large τ and by least-square fitting one can read off μ_0. Note that g is constant if and only if λ_0 is real.

To calculate real parts of other eigenvalues we use the orthogonalization method described in [3]. Here we only sketch the basic idea. Take linearly independent initial data ϕ and ψ. Let P_ϕ denote the orthogonal projection on $\{\phi\}^\perp$. One can readily show that $P_{S(\tau)\phi}S(\tau)P_\phi = P_{S(\tau)\phi}S(\tau)$. Now we define a third function $\chi := \psi + \alpha\phi$ where α is some constant. Clearly we have $P_\phi\chi = P_\phi\psi$. Suppose we choose the constant α in such a way that χ has the expansion

$$\chi = 2\sum_{k=1}^{\infty} \mathrm{Re}(c_k \varphi_k). \qquad (26)$$

Note that the sum starts at $k = 1$! For simplicity we assume that λ_0 and λ_1 are real. Then, if $\lambda_0 > \lambda_1 > \mathrm{Re}\lambda_2$,

$$S(\tau)\chi = 2e^{\lambda_1 \tau} \mathrm{Re}(c_1 \varphi_1) + \ldots \qquad (27)$$

Unfortunately we do not know the right α which is required to obtain expansion (26). But surprisingly this is not necessary since

$$P_{S(\tau)\phi}S(\tau)\psi = P_{S(\tau)\phi}S(\tau)P_\phi\psi = P_{S(\tau)\phi}S(\tau)P_\phi\chi =$$

$$P_{S(\tau)\phi}S(\tau)\chi = e^{\lambda_1 \tau}P_{S(\tau)\phi}\mathrm{Re}(c_1 \varphi_1) + \ldots \qquad (28)$$

Observe that for real λ_0 we have

$$P_{S(\tau)\phi} \sim P_{\mathrm{Re}(c_0 \varphi_0)} \qquad (29)$$

for large τ. It follows that

$$\|P_{S(\tau)\phi}S(\tau)\psi\| \sim e^{\lambda_1 \tau} \qquad (30)$$

for large τ and we can read off λ_1. In practical application the orthogonal projection is performed at every timestep (see [3] for details). The probably most important feature of this method is that it works with an arbitrary inner product. The eigenfunctions φ_k do not have to be orthogonal.

3.2 Calculating eigenvalues by shooting

Another possiblity for calculating eigenvalues is to use eq. (15). One has to solve an ODE boundary value problem where the boundary conditions are given by the regularity requirements (16). This problem can be solved using a standard shooting to a fitting point technique. Although this method is very accurate it has at least two serious disadvantages:

- It is possible to "miss" some eigenvalues because one has to start with sufficiently good initial guesses of the shooting parameters.

- The method cannot be expected to work for large negative eigenvalues (see [4]).

Nevertheless it is a good check for results obtained by time evolution.

4 Numerical results

We have calculated four eigenvalues of the ground state and the first excitation using the orthogonalization method. The first negative eigenvalue can be obtained by shooting as well. The positive eigenvalues have been computed before in [2] via shooting. Bizoń [4] has also calculated a large number of negative eigenvalues for the ground state using a completely different method. His results are in good agreement with ours. Furthermore, in [4] it has been shown analytically that $\lambda = -2$ is an eigenvalue of the ground state.

The eigenvalue $\lambda = 1$ corresponds to a so-called *gauge mode* which is associated with the freedom of choosing the blow-up time T in the coordinate transformation (9) (see [2] for a more detailed explanation). This gauge mode has no physical relevance.

We summarize our results in the following two tables.

Eigenmode	Time Evolution	Shooting
Gauge	1	1
First stable	-0.5424	-0.54246
Second stable	-2.00	
Third stable	-3.3	

Table 1: Eigenvalues of the ground state

Eigenmode	Time Evolution	Shooting
Unstable	6.3336	6.333625
Gauge	1	1
First stable	-0.518	-0.5186
Second stable	-1.7	

Table 2: Eigenvalues of the first excitation

Remarkably we do not see any oscillations and therefore all eigenvalues seem to be real.

5   Acknowledgments

A   Numerical integration of the evolution equation

A.1   Integration scheme

For integrating eq. (17) we apply the *second-order characteristic method* used e.g. in [5]. We write eq. (17) as

$$\frac{\partial u}{\partial \tau} = A\frac{\partial u}{\partial \rho} + Bu \tag{31}$$

where A and B are 3×3 matrices depending on ρ. The canonical decomposition of the matrix A is given by

$$A = \lambda_- A_- + \lambda_+ A_+ \tag{32}$$

where λ_-, λ_+ are the two non-zero eigenvalues of A. A_- and A_+ are the orthogonal projections on the eigenspaces. The (signs of the) eigenvalues of A define the directions of the characteristics of the system (31). We discretize the system (31) in space by

$$\frac{\partial u}{\partial \tau} = \lambda_- A_- D(\lambda_-)u + \lambda_+ A_+ D(\lambda_+)u + Bu \tag{33}$$

where $D(\lambda)$ is a finite difference operator producing either 3-point left-sided derivatives or 3-point right-sided derivatives depending on the sign of λ. For the discretization in time we use a Runge-Kutta-like scheme (cf. [5]).

A.2   Boundary conditions

At the boundary $\rho = 1$ we do not have to impose any boundary condition because it turns out that $\lambda_\pm \leq 0$ for $\rho \geq 1$. Therefore we do not need right-sided derivatives here. Physically this is clear since $\rho = 1$ is the past lightcone of the singularity and therefore no information can come in from outside. At $\rho = 1 - \Delta\rho$ (where $\Delta\rho$ denotes the mesh of our grid) we do need a right-sided derivative and since we are using 3-point approximations we need one point outside the lightcone $\rho = 1$. Therefore we integrate the equation on $\rho \in [0, 1 + \Delta\rho]$.

At the center $\rho = 0$ we have the regularity conditions (cf. (13))

$$u_1(\tau, 0) \equiv 0 \text{ and } \frac{\partial u_3}{\partial \rho}(\tau, 0) \equiv 0. \tag{34}$$

To impose these boundary conditions we define ghost points by

$$u_1(\tau, -\rho) = -u_1(\tau, \rho) \tag{35}$$

and use the same integration scheme as in the interior.

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